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Research Article

Nonhomogeneous Nonlinear Dirichlet Problems with a p-Superlinear Reaction

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We consider a nonlinear Dirichlet elliptic equation driven by a nonhomogeneous differential operator and with a Carathéodory reaction $f(z,\zeta)$, whose primitive $f(z,\zeta)$ is p-superlinear near $\pm\infty$, but need not satisfy the usual in such cases, the Ambrosetti-Rabinowitz condition. Using a combination of variational methods with the Morse theory (critical groups), we show that the problem has at least three nontrivial smooth solutions. Our result unifies the study of "superlinear" equations monitored by some differential operators of interest like the p-Laplacian, the (p,q)-Laplacian, and the p-generalized mean curvature operator.

1. Introduction

The motivation for this paper comes from the work of Wang [1] on superlinear Dirichlet equations. More precisely, let $\Omega \subseteq \mathbb{R}^N$ be a domain with a C^2-boundary $\partial\Omega$. Wang [1] studied the following Dirichlet problem:

$$\begin{aligned} -\Delta u(z) &= f(u(z)) \quad \text{in } \Omega, \\ u|_{\partial\Omega} &= 0. \end{aligned} \tag{1.1}$$

He assumes that $f \in C^1(\mathbb{R})$, $f(0) = f'(0) = 0$, $|f'(\zeta)| \le c(1 + |\zeta|^{r-2})$, with $1 < r < 2^*$, where

$$2^* = \begin{cases} \dfrac{2N}{N-2} & \text{if } N \ge 3, \\ +\infty & \text{if } N = 1,2 \end{cases} \tag{1.2}$$

and that there exist $\mu > 2$ and $M > 0$, such that

$$0 < \mu F(\zeta) \leq f(\zeta)\zeta \quad \forall |\zeta| \geq M, \tag{1.3}$$

where

$$F(\zeta) = \int_0^{\zeta} f(s)ds \tag{1.4}$$

(this is the so-called Ambrosetti-Rabinowitz condition). Under these hypotheses, Wang [1] proved that problem (1.1) has at least three nontrivial solutions.

The aim of this work is to establish the result of Wang [1] for a larger class of nonlinear Dirichlet problems driven by a nonhomogeneous nonlinear differential operator. In fact, our formulation unifies the treatment of "superlinear" equations driven by the p-Laplacian, the (p,q)-Laplacian, and the p-generalized mean curvature operators. In addition, our reaction term $f(z,\zeta)$ is z dependent, need not be C^1 in the ζ-variable, and in general does not satisfy the Ambrosetti-Rabinowitz condition. Instead, we employ a weaker "superlinear" condition, which incorporates in our framework functions with "slower" growth near $\pm\infty$. An earlier extension of the result of Wang [1] to equations driven by the p-Laplacian was obtained by Jiang [2, Theorem 12, p.1236] with a continuous "superlinear" reaction $f(z,\zeta)$ satisfying the Ambrosetti-Rabinowitz condition.

So, let $\Omega \subseteq \mathbb{R}^N$ be as above. The problem under consideration is the following:

$$\begin{gathered} -\operatorname{div} a(\nabla u(z)) = f(z,u(z)) \quad \text{a.e. in } \Omega, \\ u|_{\partial\Omega} = 0. \end{gathered} \tag{1.5}$$

Here $a : \mathbb{R}^N \to \mathbb{R}^N$ is a map which is strictly monotone and satisfies certain other regularity conditions. The precise conditions on a are formulated in hypotheses $H(a)$. These hypotheses are rather general, and as we already mentioned, they unify the treatment of various differential operators of interest. The reaction $f(z,\zeta)$ is a Carathéodory function (i.e., for all $\zeta \in \mathbb{R}$, the function $z \mapsto f(z,\zeta)$ is measurable and for almost all $z \in \Omega$, the function $\zeta \mapsto f(z,\zeta)$ is continuous). We assume that the primitive

$$F(z,\zeta) = \int_0^{\zeta} f(z,s)ds \tag{1.6}$$

exhibits p-superlinear growth near $\pm\infty$. However, we do not employ the usual in such cases, the Ambrosetti-Rabinowitz condition. Instead we use a weaker condition (see hypotheses $H(f)$), which permits the consideration of a broader class of reaction terms.

Our approach is variational based on the critical point theory combined with Morse theory (critical groups). In the next section for easy reference, we present the main mathematical tools that we will use in the paper. We also state the precise hypotheses on the maps a and f and explore some useful consequences of them.

2. Mathematical Background and Hypotheses

Let X be a Banach space, and let X^* be its topological dual. By $\langle\cdot,\cdot\rangle$ we denote the duality brackets for the pair (X^*, X). Let $\varphi \in C^1(X)$. We say that φ satisfies the *Cerami condition* if the following is true:

"every sequence $\{x_n\}_{n\geq 1} \subseteq X$, such that $\{\varphi(x_n)\}_{n\geq 1}$ is bounded and

$$(1 + \|x_n\|)\varphi'(x_n) \longrightarrow 0 \text{ in } X^*, \tag{2.1}$$

admits a strongly convergent subsequence."

This compactness-type condition is in general weaker than the usual Palais-Smale condition. Nevertheless, the Cerami condition suffices to have a deformation theorem, and from it the minimax theory of certain critical values of φ is derive (see, e.g., Gasiński and Papageorgiou [3]). In particular, we can state the following theorem, known in the literature as the "mountain pass theorem."

Theorem 2.1. *If $\varphi \in C^1(X)$ satisfies the Cerami condition, $x_0, x_1 \in X$ are such that $\|x_1 - x_0\| > \varrho > 0$, and*

$$\begin{gathered} \max\{\varphi(x_0),\ \varphi(x_1)\} < \inf\{\varphi(x) : \|x - x_0\| = \varrho\} = \eta_\varrho, \\ c = \inf_{\gamma\in\Gamma} \max_{0\leq t\leq 1} \varphi(\gamma(t)), \end{gathered} \tag{2.2}$$

where

$$\Gamma = \{\gamma \in C([0,1]; X) : \gamma(0) = x_0,\ \gamma(1) = x_1\}, \tag{2.3}$$

then $c \geq \eta_\varrho$ and c is a critical value of φ.

In the analysis of problem (1.5) in addition to the Sobolev space $W_0^{1,p}(\Omega)$, we will also use the Banach space

$$C_0^1\left(\overline{\Omega}\right) = \left\{u \in C^1\left(\overline{\Omega}\right) : u|_{\partial\Omega} = 0\right\}. \tag{2.4}$$

This is an ordered Banach space with positive cone

$$C_+ = \left\{u \in C_0^1\left(\overline{\Omega}\right) :\ u(z) \geq 0\ \forall z \in \overline{\Omega}\right\}. \tag{2.5}$$

This cone has a nonempty interior, given by

$$\operatorname{int} C_+ = \left\{u \in C_+ :\ u(z) > 0\ \forall z \in \Omega,\ \frac{\partial u}{\partial n}(z) < 0\ \forall z \in \partial\Omega\right\}, \tag{2.6}$$

where $n(\cdot)$ denotes the outward unit normal on $\partial\Omega$.

In what follows, by $\widehat{\lambda}_1$ we denote the first eigenvalue of $(-\Delta_p, W_0^{1,p}(\Omega))$, where Δ_p denotes the p-Laplace operator, defined by

$$\Delta_p u = \operatorname{div}\left(\|\nabla u\|^{p-2}\nabla u\right) \quad \forall u \in W_0^{1,p}(\Omega). \tag{2.7}$$

We know (see, e.g., Gasiński and Papageorgiou [3]) that $\widehat{\lambda}_1 > 0$ is isolated and simple (i.e., the corresponding eigenspace is one-dimensional) and

$$\widehat{\lambda}_1 = \inf\left\{ \frac{\|\nabla u\|_p^p}{\|u\|_p^p} : u \in W_0^{1,p}(\Omega),\ u \neq 0 \right\}. \tag{2.8}$$

In this variational characterization of $\widehat{\lambda}_1$, the infimum is realized on the corresponding one-dimensional eigenspace. From (2.8), we see that the elements of the eigenspace do not change sign. In what follows, by $\widehat{u}_1$ we denote the L^p-normalized (i.e., $\|\widehat{u}_1\|_p = 1$) positive eigenfunction corresponding to $\widehat{\lambda}_1 > 0$. The nonlinear regularity theory for the p-Laplacian equations (see, e.g., Gasiński and Papageorgiou [3, p. 737]) and the nonlinear maximum principle of Vázquez [4] imply that $\widehat{u}_1 \in \operatorname{int} C_+$.

Now, let $\varphi \in C^1(X)$ and let $c \in \mathbb{R}$. We introduce the following notation:

$$\begin{aligned} \varphi^c &= \{x \in X : \varphi(x) \leq c\}, \\ K_\varphi &= \{x \in X : \varphi'(x) = 0\}, \\ K_\varphi^c &= \{x \in K_\varphi : \varphi(x) = c\}. \end{aligned} \tag{2.9}$$

Let (Y_1, Y_2) be a topological pair with $Y_2 \subseteq Y_1 \subseteq X$. For every integer $k \geq 0$, by $H_k(Y_1, Y_2)$ we denote the kth relative singular homology group for the pair (Y_1, Y_2) with integer coefficients. The critical groups of φ at an isolated point $x_0 \in K_\varphi$ with $\varphi(x_0) = c$ (i.e., $x_0 \in K_\varphi^c$) are defined by

$$C_k(\varphi, x_0) = H_k(\varphi^c \cap U,\ \varphi^c \cap U \setminus \{x_0\}) \quad \forall k \geq 0, \tag{2.10}$$

where U is a neighbourhood of x_0, such that $K_\varphi \cap \varphi^c \cap U = \{x_0\}$. The excision property of singular homology implies that this definition is independent of the particular choice of the neighbourhood U.

Suppose that $\varphi \in C^1(X)$ satisfies the Cerami condition and $\varphi(K_\varphi) > -\infty$. Let $c < inf\varphi(K_\varphi)$. The critical groups of φ at infinity are defined by

$$C_k(\varphi, \infty) = H_k(X, \varphi^c) \quad \forall k \geq 0. \tag{2.11}$$

The second deformation theorem (see, e.g., Gasiński and Papageorgiou [3, p. 628]) guarantees that this definition is independent of the particular choice of the level $c < inf\varphi(K_\varphi)$.

Suppose that K_φ is finite. We set

$$M(t,x) = \sum_{k\geq 0} \operatorname{rank} C_k(\varphi,x)t^k \quad \forall t\in\mathbb{R},\ x\in K_\varphi,$$
$$P(t,\infty) = \sum_{k\geq 0} \operatorname{rank} C_k(\varphi,\infty)t^k \quad \forall t\in\mathbb{R}. \tag{2.12}$$

The Morse relation says that

$$\sum_{x\in K_\varphi} M(t,x) = P(t,\infty) + (1+t)Q(t), \tag{2.13}$$

where

$$Q(t) = \sum_{k\geq 0}\beta_k t^k \tag{2.14}$$

is a formal series in $t\in\mathbb{R}$ with nonnegative integer coefficients $\beta_k\in\mathbb{N}$.

Now we will introduce the hypotheses on the maps $a(y)$ and $f(z,\zeta)$. So, let $h\in C^1(0,+\infty)$ be such that

$$0 < \frac{th'(t)}{h(t)} \leq c_0 \quad \forall t>0, \tag{2.15}$$

for some $c_0>0$ and

$$c_1 t^{p-1} \leq h(t) \leq c_2\left(1+|t|^{p-1}\right) \quad \forall t>0, \tag{2.16}$$

for some $c_1, c_2 > 0$.

The hypotheses on the map $a(y)$ are the following:

$\underline{H(a)}$: $a(y) = a_0(\|y\|)y$, where $a_0(t)>0$ for all $t>0$ and

(i) $a\in C(\mathbb{R}^N;\mathbb{R}^N)\cap C^1(\mathbb{R}^N\setminus\{0\};\mathbb{R}^N)$,

(ii) there exists $c_3>0$, such that

$$\|\nabla a(y)\| \leq c_3\frac{h(\|y\|)}{\|y\|} \quad \forall y\in\mathbb{R}^N\setminus\{0\}, \tag{2.17}$$

(iii) we have

$$(\nabla a(y)\xi,\ \xi)_{\mathbb{R}^N} \geq \frac{h(\|y\|)}{\|y\|}\|\xi\|^2 \quad \forall y\in\mathbb{R}^N\setminus\{0\},\ \xi\in\mathbb{R}^N, \tag{2.18}$$

(iv) if $G : \mathbb{R}^N \to \mathbb{R}^N$ is a function, such that $\nabla G(y) = a(y)$ for $y \in \mathbb{R}^N$ and $G(0) = 0$, then there exists $c_4 > 0$, such that

$$pG(y) - (a(y), y)_{\mathbb{R}^N} \geq -c_4 \quad \forall y \in \mathbb{R}^N. \tag{2.19}$$

Remark 2.2. Let

$$G_0(t) = \int_0^t a_0(s)s\,ds \quad \forall t \geq 0. \tag{2.20}$$

Evidently G_0 is strictly convex and strictly increasing on $\mathbb{R}_+ = [0, +\infty)$. We set

$$G(y) = G_0(\|y\|) \quad \forall y \in \mathbb{R}^N. \tag{2.21}$$

Then G is convex, $G(0) = 0$, and

$$\nabla G(y) = G_0'(\|y\|)\frac{y}{\|y\|} = a_0(\|y\|)y = a(y) \quad \forall y \in \mathbb{R}^N \setminus \{0\}. \tag{2.22}$$

Hence the primitive function $G(y)$ used in hypothesis $H(a)(iv)$ is uniquely defined. Note that the convexity of G implies that

$$G(y) \leq (a(y), y)_{\mathbb{R}^N} \quad \forall y \in \mathbb{R}^N. \tag{2.23}$$

Hypotheses $H(a)$ and (2.23) lead easily to the following lemma summarizing the main properties of a.

Lemma 2.3. *If hypotheses $H(a)$ hold, then*

(a) *the map $y \mapsto a(y)$ is maximal monotone and strictly monotone,*

(b) *there exists $c_5 > 0$, such that*

$$\|a(y)\| \leq c_5\left(1 + \|y\|^{p-1}\right) \quad \forall y \in \mathbb{R}^N, \tag{2.24}$$

(c) *we have*

$$(a(y), y)_{\mathbb{R}^N} \geq \frac{c_1}{p-1}\|y\|^p \quad \forall y \in \mathbb{R}^N \tag{2.25}$$

(where $c_1 > 0$ is as in (2.16)*).*

From the above lemma and the integral form of the mean value theorem, we have the following result.

Corollary 2.4. *If hypotheses $H(a)$ hold, then there exists $c_6 > 0$, such that*

$$\frac{c_1}{p(p-1)}\|y\|^p \le G(y) \le c_6\left(1+\|y\|^p\right) \quad \forall y \in \mathbb{R}^N. \tag{2.26}$$

Example 2.5. The following maps satisfy hypotheses $H(a)$:

(a) $a(y) = \|y\|^{p-2}y$ with $1 < p < +\infty$.

This map corresponds to the p-Laplace differential operator

$$\Delta_p u = \operatorname{div}\left(\|\nabla u\|^{p-2}\nabla u\right) \quad \forall u \in W_0^{1,p}(\Omega). \tag{2.27}$$

(b) $a(y) = \|y\|^{p-2}y + \mu\|y\|^{p-2}y$ with $2 \le q < p < +\infty,\ \mu \ge 0$.

This map corresponds to the (p,q)-Laplace differential operator

$$\Delta_p u + \mu\Delta_q u \quad \forall u \in W_0^{1,p}(\Omega). \tag{2.28}$$

This is an important operator occurring in quantum physics (see Benci et al. [5]). Recently there have been some papers dealing with the existence and multiplicity of solutions for equations driven by such operators. We mention the works of Cingolani and Degiovanni [6], Figueiredo [7], and Sun [8]:

(c) $a(y) = \left(1+\|y\|^2\right)^{(p-2)/2}y$, with $2 \le p < +\infty$.

This map corresponds to the p-generalized mean curvature operator

$$\operatorname{div}\left(\left(1+\|\nabla u\|^2\right)^{(p-2)/2}\nabla u\right) \quad \forall u \in W_0^{1,p}(\Omega). \tag{2.29}$$

Such equations were investigated by Chen-Shen [9]:

(d) $a(y) = \|y\|^{p-2}y + \|y\|^{p-2}y/(1+\|y\|^p)$, with $1 < p < +\infty$,

(e) $a(y) = \|y\|^{p-2}y + \ln(1+\|y\|^{p-2})y$, with $2 \le p < +\infty$.

Let $A : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega) = W_0^{1,p}(\Omega)^*$ (with $1/p + 1/p' = 1$) be the nonlinear map, defined by

$$\langle A(u), y\rangle = \int_\Omega \left(a(\nabla u(z)), \nabla y(z)\right)_{\mathbb{R}^N} dz \quad \forall u, y \in W_0^{1,p}(\Omega). \tag{2.30}$$

From Gasiński and Papageorgiou [10, Proposition 3.1, p. 852], we have the following result for this map.

Proposition 2.6. *If hypotheses $H(a)$ hold, then the map $A : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ defined by* (2.30) *is bounded, continuous, strictly monotone, hence maximal monotone too, and of type $(S)_+$; that is, if $u_n \to u$ weakly in $W_0^{1,p}(\Omega)$ and*

$$\limsup_{n\to+\infty}\langle A(u_n),\ u_n-u\rangle \le 0, \tag{2.31}$$

then $u_n \to u$ in $W_0^{1,p}(\Omega)$.

The next lemma is an easy consequence of (2.8) and of the fact that $\widehat{u}_1 \in \operatorname{int} C_+$ (see, e.g., Papageorgiou and Kyritsi-Yiallourou [11, p. 356]).

Proposition 2.7. *If $\vartheta \in L^\infty(\Omega)_+$, $\vartheta(z) \le (c_1/p(p-1))\widehat{\lambda}_1$ for almost all $z \in \Omega$, $\vartheta \neq (c_1/p(p-1))\widehat{\lambda}_1$, then there exists $\xi_0 > 0$, such that*

$$\frac{c_1}{p-1}\|\nabla u\|_p^p - \int_\Omega \vartheta(z)|u(z)|^p\,dz \ge \xi_0\|\nabla u\|_p^p \quad \forall u \in W_0^{1,p}(\Omega). \tag{2.32}$$

The hypotheses on the reaction $f(z,\zeta)$ are the following:

$\underline{H(f)}$: $f : \Omega\times\mathbb{R} \to \mathbb{R}$ is a Carathéodory function, such that $f(z,0) = 0$ for almost all $z \in \Omega$ and

(i) *there exist $a \in L^\infty(\Omega)_+$, $c > 0$ and $r \in (p,p^*)$, with*

$$p^* = \begin{cases} \dfrac{Np}{N-p} & \text{if } p < N, \\ +\infty & \text{if } p \ge N, \end{cases} \tag{2.33}$$

such that

$$\left|f(z,\zeta)\right| \le a(z) + c|\zeta|^{r-1} \quad \text{for almost all } z \in \Omega, \quad \text{all } \zeta \in \mathbb{R}; \tag{2.34}$$

(ii) *if*

$$F(z,\zeta) = \int_0^\zeta f(z,s)ds, \tag{2.35}$$

then

$$\lim_{\zeta\to\pm\infty}\frac{F(z,\zeta)}{|\zeta|^p} = +\infty \quad \text{uniformly for almost all } z \in \Omega, \tag{2.36}$$

(iii) *there exist $\tau \in ((r-p)\max\{1,N/p\},p^*)$ and $\beta_0 > 0$, such that*

$$\liminf_{\zeta\to\pm\infty}\frac{f(z,\zeta)\zeta - pF(z,\zeta)}{|\zeta|^\tau} \ge \beta_0 \quad \text{uniformly for almost all } z \in \Omega, \tag{2.37}$$

(iv) *there exists* $\vartheta \in L^\infty(\Omega)_+$, *such that* $\vartheta(z) \leq (c_1/p(p-1))\widehat{\lambda}_1$ *for almost all* $z \in \Omega$, $\vartheta \neq (c_1/p(p-1))\widehat{\lambda}_1$, *and*

$$\limsup_{\zeta\to 0}\frac{pF(z,\zeta)}{|\zeta|^p}\leq \vartheta(z) \quad \text{uniformly for almost all } z\in\Omega; \tag{2.38}$$

(v) *for every* $\varrho > 0$, *there exists* $\xi_\varrho > 0$, *such that*

$$f(z,\zeta)\zeta + \xi_\varrho|\zeta|^p \geq 0 \quad \text{for almost all } z\in\Omega,\ \text{all } |\zeta|\leq\varrho. \tag{2.39}$$

Remark 2.8. Hypothesis $H(f)$(ii) implies that for almost all $z \in \Omega$, the primitive $F(z,\cdot)$ is p-superlinear. Evidently, this condition is satisfied if

$$\lim_{\zeta\to\pm\infty}\frac{f(z,\zeta)}{|\zeta|^{p-2}\zeta} = +\infty \quad \text{uniformly for almost all } z\in\Omega, \tag{2.40}$$

that is, for almost all $z \in \Omega$, the reaction $f(z,\cdot)$ is $(p-1)$-superlinear. However, we do not use the usual for "superlinear" problems, the Ambrosetti-Rabinowitz condition. We recall that this condition says that there exist $\mu > 0$ and $M > 0$, such that

$$\begin{gathered} 0<\mu F(z,\zeta)\leq f(z,\zeta)\zeta \quad \text{for almost all } z\in\Omega,\ \text{all } |\zeta|\geq M,\\ \operatorname*{ess\,sup}_{\Omega} F(\cdot,M)>0. \end{gathered} \tag{2.41}$$

Integrating (2.41), we obtain the weaker condition

$$c_7|\zeta|^\mu \leq F(z,\zeta) \quad \text{for almost all } z\in\Omega,\ \text{all } |\zeta|\geq M, \tag{2.42}$$

with $c_7 > 0$. Evidently from (2.42) we have the much weaker condition

$$\lim_{\zeta\to\pm\infty}\frac{F(z,\zeta)}{|\zeta|^p} = +\infty \quad \text{uniformly for almost all } z\in\Omega. \tag{2.43}$$

Here, the p-superlinearity condition (2.43) is coupled with hypothesis $H(f)$(iii), which is weaker than the Ambrosetti-Rabinowitz condition (2.41). Indeed, suppose that the Ambrosetti-Rabinowitz condition is satisfied. We may assume that $\mu > (r-p)\max\{1, N/p\}$. Then

$$\frac{f(z,\zeta)\zeta - pF(z,\zeta)}{|\zeta|^\mu} = \frac{f(z,\zeta)\zeta-\mu F(z,\zeta)}{|\zeta|^\mu} + \frac{(\mu-p)F(z,\zeta)}{|\zeta|^\mu} \geq (\mu-p)c_7 \tag{2.44}$$

(see (2.41) and (2.42)). Therefore, hypothesis $H(f)$(iii) is satisfied.

Example 2.9. The following function satisfies hypotheses $H(f)$, but not the Ambrosetti-Rabinowitz condition. For the sake of simplicity we drop the z-dependence:

$$f(\zeta) = \begin{cases} |\zeta|^{q-2}\zeta & \text{if } |\zeta| \le 1, \\ p|\zeta|^{p-2}\zeta\left(\ln|\zeta| + \dfrac{1}{p}\right) & \text{if } |\zeta| > 1, \end{cases} \tag{2.45}$$

with $1 < p < q < +\infty$.

In this work, for every $u \in W_0^{1,p}(\Omega)$, we set

$$\|u\| = \|\nabla_u\|_p \tag{2.46}$$

(by virtue of the Poincaré inequality). We mention that the notation $\|\cdot\|$ will also be used to denote the $\mathbb{R}^N$-norm. However, no confusion is possible, since it is always clear from the context, whose norm is used. For every $\zeta \in \mathbb{R}$, we set

$$\zeta^{\pm} = \max\{\pm\zeta, 0\}, \tag{2.47}$$

and for $u \in W_0^{1,p}(\Omega)$, we define

$$u^{\pm}(\cdot) = u(\cdot)^{\pm}. \tag{2.48}$$

Then $u^{\pm} \in W_0^{1,p}(\Omega)$, and we have

$$u = u^+ - u^-, \qquad |u| = u^+ + u^-. \tag{2.49}$$

By $|\cdot|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$. Finally, for a given measurable function $h : \Omega \times \mathbb{R} \to \mathbb{R}$ (e.g., a Carathéodory function), we define

$$N_h(u)(\cdot) = h(\cdot, u(\cdot)) \quad \forall u \in W_0^{1,p}(\Omega) \tag{2.50}$$

(the Nemytskii map corresponding to $h(\cdot,\cdot)$).

3. Three-Solution Theorem

In this section, we prove a multiplicity theorem for problem (1.5), producing three nontrivial smooth solutions, two of which have constant sign (one positive, the other negative).

First we produce two constant sign solutions of (1.5). For this purpose, we introduce the positive and negative truncations of $f(z,\cdot)$, namely:

$$f_{\pm}(z,\zeta) = f\left(z, \pm\zeta^{\pm}\right) \quad \forall (z,\zeta) \in \Omega \times \mathbb{R}. \tag{3.1}$$

Both are Carathéodory functions. We set

$$F_{\pm}(z,\zeta) = \int_0^{\zeta} f_{\pm}(z,s)ds \tag{3.2}$$

and consider the C^1-functionals $\varphi_{\pm} : W_0^{1,p}(\Omega) \to \mathbb{R}$, defined by

$$\varphi_{\pm}(u) = \int_{\Omega} G(\nabla u(z))dz - \int_{\Omega} F_{\pm}(z,u(z))dz \quad \forall u \in W_0^{1,p}(\Omega). \tag{3.3}$$

Also, let $\varphi : W_0^{1,p}(\Omega) \to \mathbb{R}$ be the C^1-energy functional for problem (1.5), defined by

$$\varphi(u) = \int_{\Omega} G(\nabla u(z))dz - \int_{\Omega} F(z,u(z))dz \quad \forall u \in W_0^{1,p}(\Omega). \tag{3.4}$$

Proposition 3.1. *If hypotheses $H(a)$ and $H(f)$ hold, then the functionals $\varphi_{\pm}$ satisfy the Cerami condition.*

Proof. We do the proof for φ_{+}, the proof for φ_{-} being similar.

So, let $\{u_n\}_{n\geq 1} \subseteq W_0^{1,p}(\Omega)$ be a sequence, such that

$$\left|\varphi_{+}(u_n)\right| \leq M_1 \quad \forall n \geq 1, \tag{3.5}$$

for some $M_1 > 0$, and

$$(1 + \|u_n\|)\varphi'_{+}(u_n) \longrightarrow 0 \quad \text{in } W^{-1,p'}(\Omega). \tag{3.6}$$

From (3.6), we have

$$\left|\langle \varphi'_{+}(u_n),\ v\rangle\right| \leq \frac{\varepsilon_n \|v\|}{1 + \|u_n\|} \quad \forall v \in W_0^{1,p}(\Omega), \tag{3.7}$$

with $\varepsilon_n \searrow 0$, so

$$\left|\langle A(u_n),\ v\rangle - \int_{\Omega} f_{+}(z,u_n)v\, dz\right| \leq \frac{\varepsilon_n \|v\|}{1 + \|u_n\|} \quad \forall n \geq 1. \tag{3.8}$$

In (3.8), first we choose $v = -u_n^{-} \in W_0^{1,p}(\Omega)$. Then using Lemma 2.3(c), we have

$$\frac{c_1}{p-1}\left\|\nabla u_n^{-}\right\|_p^p \leq \varepsilon_n \quad \forall n \geq 1, \tag{3.9}$$

so

$$u_n^{-} \longrightarrow 0 \quad \text{in } W_0^{1,p}(\Omega). \tag{3.10}$$

Next, in (3.8) we choose $v = u_n^+ \in W_0^{1,p}(\Omega)$. We obtain

$$-\int_\Omega (a(\nabla u_n^+),\ \nabla u_n^+)_{\mathbb{R}^N} + \int_\Omega f(z, u_n^+)u_n^+\, dz \le \varepsilon_n \quad \forall n \ge 1. \tag{3.11}$$

From (3.5) and (3.10), we have

$$\int_\Omega pG(\nabla u_n^+)dz - \int_\Omega pF(z, u_n^+)dz \le M_2 \quad \forall n \ge 1 \tag{3.12}$$

for some $M_2 > 0$. Adding (3.11) and (3.12), we obtain

$$\int_\Omega (pG(\nabla u_n^+) - (a(\nabla u_n^+), \nabla u_n^+)_{\mathbb{R}^N})dz + \int_\Omega (f(z, u_n^+)u_n^+ - pF(z, u_n^+))dz \le M_3 \quad \forall n \ge 1, \tag{3.13}$$

for some $M_3 > 0$, so

$$\int_\Omega (f(z, u_n^+)u_n^+ - pF(z, u_n^+))dz \le M_4 = M_3 + c_4|\Omega|_N \quad \forall n \ge 1 \tag{3.14}$$

(see hypothesis $H(a)$(iv)).

Hypotheses $H(f)$(i) and (iii) imply that we can find $\beta_1 \in (0, \beta_0)$ and $c_8 > 0$, such that

$$\beta_1|\zeta|^\tau - c_8 \le f(z,\zeta)\zeta - pF(z,\zeta) \quad \text{for almost all } z \in \Omega,\ \text{all } \zeta \in \mathbb{R}. \tag{3.15}$$

Using (3.15) and (3.14), we obtain

$$\beta_1\|u_n^+\|_\tau^\tau \le M_5 \quad \forall n \ge 1, \tag{3.16}$$

with $M_5 = M_4 + c_8|\Omega|_N > 0$ and so

$$\text{the sequence } \{u_n^+\}_{n\ge1} \subseteq L^\tau(\Omega) \text{ is bounded.} \tag{3.17}$$

First suppose that $N \ne p$. From hypothesis $H(f)$(iii), it is clear that we can always assume that $\tau \le r < p^*$. So, we can find $t \in [0,1)$, such that

$$\frac{1}{r} = \frac{1-t}{\tau} + \frac{t}{p^*}. \tag{3.18}$$

Invoking the interpolation inequality (see, e.g., Gasiński and Papageorgiou [3, p. 905]), we have

$$\|u_n^+\|_r \le \|u_n^+\|_\tau^{1-t}\|u_n^+\|_{p^*}^t, \tag{3.19}$$

so

$$\|u_n^+\|_r^r \leq M_6 \|u_n^+\|^{tr} \quad \forall n \geq 1, \tag{3.20}$$

for some $M_6 > 0$ (see (3.17) and use the Sobolev embedding theorem).

Recall that

$$\left| \int_\Omega (a(\nabla u_n^+), \nabla u_n^+)_{\mathbb{R}^N} dz - \int_\Omega f(z, u_n^+) u_n^+ \, dz \right| \leq \varepsilon_n \quad \forall n \geq 1. \tag{3.21}$$

From hypothesis $H(f)$(i), we have

$$f(z,\zeta)\zeta \leq \hat{a}(z) + \hat{c}|\zeta|^r \quad \text{for almost all } z \in \Omega, \text{ all } \zeta \in \mathbb{R}, \tag{3.22}$$

with $\hat{a} \in L^\infty(\Omega)_+$, $\hat{c} > 0$. Therefore, from (3.21) and Lemma 2.3(c), we have

$$\frac{c_1}{p-1} \|\nabla u_n^+\|_p^p \leq c_9 \left(1 + \|u_n^+\|_r^r\right) \quad \forall n \geq 1, \tag{3.23}$$

for some $c_9 > 0$ and so

$$\|u_n^+\|^p \leq c_{10} \left(1 + \|u_n^+\|^{tr}\right) \quad \forall n \geq 1, \tag{3.24}$$

for some $c_{10} > 0$ (see (3.20)). The hypothesis on τ (see $H(f)$(iii)) implies that $tr < p$, and so

$$\text{the sequence } \{u_n^+\}_{n \geq 1} \subseteq W_0^{1,p}(\Omega) \text{ is bounded,} \tag{3.25}$$

and thus

$$\text{the sequence } \{u_n\}_{n \geq 1} \subseteq W_0^{1,p}(\Omega) \text{ is bounded} \tag{3.26}$$

(see (3.26)).

Now, suppose that $N = p$. In this case, we have $p^* = +\infty$, while from the Sobolev embedding theorem, we have that $W_0^{1,p}(\Omega) \subseteq L^q(\Omega)$ for all $q \in [1, +\infty)$. So, we need to modify the previous argument. Let $\vartheta > r \geq \tau$. Then we choose $t \in [0,1)$, such that

$$\frac{1}{r} = \frac{1-t}{\tau} + \frac{t}{\vartheta}, \tag{3.27}$$

so

$$tr = \frac{\vartheta(r-\tau)}{\vartheta - \tau}. \tag{3.28}$$

Note that

$$\frac{\vartheta(r-\tau)}{\vartheta-\tau} \longrightarrow r-\tau \quad \text{as } \vartheta \longrightarrow +\infty = p^*. \tag{3.29}$$

Since $N = p$, we have $r - \tau < p$ (see $H(f)(iii)$). Therefore, for large $\vartheta > r$, we have that $tr < p$ (see (3.28)). Hence, if in the previous argument, we replace p^* with such a large $\vartheta > r$, again we reach (3.26).

Because of (3.26), we may assume that

$$u_n \longrightarrow u \quad \text{weakly in } W_0^{1,p}(\Omega), \tag{3.30}$$

$$u_n \longrightarrow u \quad \text{in } L^p(\Omega). \tag{3.31}$$

In (3.8), we choose $v = u_n - u \in W_0^{1,p}(\Omega)$, pass to the limit as $n \to +\infty$, and use (3.30). Then

$$\liminf_{n \to +\infty} \langle A(u_n),\ u_n - u\rangle \ = 0, \tag{3.32}$$

so

$$u_n \longrightarrow u \quad \text{in } W_0^{1,p}(\Omega) \tag{3.33}$$

(see Proposition 2.6). This proves that φ_+ satisfies the Cerami condition.

Similarly we show that φ_- satisfies the Cerami condition. □

With some obvious minor modifications in the above proof, we can also have the following result.

Proposition 3.2. *If hypotheses $H(a)$ and $H(f)$ hold, then the functional φ satisfies the Cerami condition.*

Next we determine the structure of the trivial critical point $u = 0$ for the functionals $\varphi_\pm$ and φ.

Proposition 3.3. *If hypotheses $H(a)$ and $H(f)$ hold, then $u = 0$ is a local minimizer for the functionals $\varphi_\pm$ and φ.*

Proof. By virtue of hypotheses $H(f)(i)$ and (iv), for a given $\varepsilon > 0$ we can find $c_\varepsilon > 0$, such that

$$F(z,\zeta) \le \frac{1}{p}(\vartheta(z)+\varepsilon)|\zeta|^p + c_\varepsilon|\zeta|^r \quad \text{for almost all } z \in \Omega,\ \text{all } \zeta \in \mathbb{R}. \tag{3.34}$$

Then for all $u \in W_0^{1,p}(\Omega)$, we have

$$\begin{aligned}
\varphi_+(u) &= \int_\Omega G(\nabla u)dz - \int_\Omega F(z,u^+)dz \\
&\geq \frac{c_1}{p(p-1)}\|\nabla u\|_p^p - \frac{1}{p}\int_\Omega \vartheta(u^+)^p \, dz - \frac{\varepsilon}{p\widehat{\lambda}_1}\|u\|^p - c_{11}\|u\|^r \\
&\geq \frac{1}{p}\left(\xi_0 - \frac{\varepsilon}{\widehat{\lambda}_1}\right)\|u\|^p - c_{11}\|u\|^r,
\end{aligned} \tag{3.35}$$

for some $c_{11} > 0$ (see Corollary 2.4, (2.8), (3.34), and Proposition 2.7). Choosing $\varepsilon \in (0, \ \widehat{\lambda}_1\xi_0)$, we have

$$\varphi_+(u) \geq c_{12}\|u\|^p - c_{11}\|u\|^r \quad \forall u \in W_0^{1,p}(\Omega), \tag{3.36}$$

for some $c_{12} > 0$. Since $r > p$, from (3.36), it follows that we can find small $\varrho \in (0,1)$, such that

$$\varphi_+(u) > 0 \quad \forall u, \ \text{with } 0 < \|u\| \leq \varrho, \tag{3.37}$$

so

$$u = 0 \text{ is a local minimizer of } \varphi_+. \tag{3.38}$$

Similarly, we show that $u = 0$ is a local minimizer for the functionals φ_- and φ. □

We may assume that $u = 0$ is an isolated critical point of φ_+ (resp., φ_-). Otherwise, we already have a sequence of distinct positive (resp., negative) solutions of (1.5) and so we are done. Moreover, as in Gasiński and Papageorgiou [12, proof of Theorem 3.4,] we can find small $\varrho_\pm \in (0,1)$, such that

$$\inf\{\varphi_\pm(u) : \|u\| = \varrho_\pm\} = \eta_\pm > 0. \tag{3.39}$$

By virtue of hypothesis $H(f)$(ii) (the p-superlinear condition), we have the next result, which completes the mountain pass geometry for problem (1.1).

Proposition 3.4. *If hypotheses $H(a)$ and $H(f)$ hold and $u \in \operatorname{int} C_+$, then $\varphi_\pm(tu) \to -\infty$ as $t \to \pm\infty$.*

Proof. By virtue of hypotheses $H(f)$(i) and (ii), for a given $\xi > 0$, we can find $c_{13} = c_{13}(\xi) > 0$, such that

$$\xi|\zeta|^p - c_{13} \leq F(z,\zeta) \quad \text{for almost all } z \in \Omega, \ \text{all } \zeta \in \mathbb{R}. \tag{3.40}$$

Then for $u \in \operatorname{int} C_+$ and $t > 0$, we have

$$\begin{aligned}\varphi_+(tu) &= \int_\Omega G(t\nabla u)dz - \int_\Omega F(z, tu)dz \\ &\leq c_{14}\left(1 + t^p \|u\|^p\right) - \xi t^p \|u\|_p^p + c_{13}|\Omega|_N \\ &= t^p \left(c_{14}\|u\|^p - \xi \|u\|_p^p\right) + c_{15},\end{aligned} \tag{3.41}$$

for some $c_{14} > 0$ and with $c_{15} = c_{14} + c_{13}|\Omega|_N > 0$ (see Corollary 2.4 and (3.40)).

Choosing $\xi > c_{14}(\|u\|^p / \|u\|_p^p)$, from (3.41), it follows that

$$\varphi_+(tu) \longrightarrow -\infty \quad \text{as } t \longrightarrow +\infty. \tag{3.42}$$

Similarly, we show that

$$\varphi_-(tu) \longrightarrow -\infty \quad \text{as } t \longrightarrow -\infty. \tag{3.43}$$

□

Now we are ready to produce two constant sign smooth solutions of (1.5).

Proposition 3.5. *If hypotheses $H(a)$ and $H(f)$ hold, then problem* (1.5) *has at least two nontrivial constant sign smooth solutions*

$$u_0 \in \operatorname{int} C_+, \qquad v_0 \in -\operatorname{int} C_+. \tag{3.44}$$

Proof. From (3.39), we have

$$\varphi_+(0) = 0 < \inf\{\varphi_+(u) : \ \|u\| = \varrho_+\} = \eta_+. \tag{3.45}$$

Moreover, according to Proposition 3.4, for $u \in \operatorname{int} C_+$, we can find large $t > 0$, such that

$$\varphi_+(tu) \leq \varphi_+(0) = 0 < \eta_+, \quad \|tu\| > \varrho_+. \tag{3.46}$$

Then because of (3.45), (3.46), and Proposition 3.1, we can apply the mountain pass theorem (see Theorem 2.1) and find $u_0 \in W_0^{1,p}(\Omega)$, such that

$$\varphi_+(0) = 0 < \eta_+ \leq \varphi_+(u_0), \tag{3.47}$$

$$\varphi'_+(u_0) = 0. \tag{3.48}$$

From (3.47) we see that $u_0 \neq 0$. From (3.48), we have

$$A(u_0) = N_{f_+}(u_0). \tag{3.49}$$

On (3.49) we act with $-u_0^- \in W_0^{1,p}(\Omega)$ and obtain

$$\frac{c_1}{p-1}\left\|\nabla u_0^-\right\|_p^p \leq 0 \tag{3.50}$$

(see Lemma 2.3(c)), so

$$u_0 \geq 0, \qquad u_0 \neq 0. \tag{3.51}$$

Then, from (3.49), we have

$$\begin{gathered} -\operatorname{div}\, a(\nabla u_0(z)) = f(z, u_0(z)) \quad \text{a.e. in } \Omega, \\ u_0|_{\partial\Omega} = 0. \end{gathered} \tag{3.52}$$

Theorem 7.1 of Ladyzhenskaya and Uraltseva [13, p. 286] implies that $u_0 \in L^\infty(\Omega)$. Then from Lieberman [14, p. 320], we have that $u_0 \in C_0^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$. Let $\varrho = \|u_0\|_\infty$, and let $\xi_\varrho > 0$ be as postulated by hypothesis $H(f)(v)$. Then

$$-\operatorname{div}\, a(\nabla u_0(z)) + \xi_\varrho u_0(z)^{p-1} \geq 0 \quad \text{for almost all } z \in \Omega \tag{3.53}$$

(see (3.52) and hypothesis $H(f)(iv)$), so

$$\operatorname{div}\, a(\nabla u_0(z)) \leq \xi_\varrho u_0(z)^{p-1} \quad \text{for almost all } z \in \Omega. \tag{3.54}$$

Then, from Theorem 5.5.1 of Pucci and Serrin [15, p. 120], we have that $u_0 \in \operatorname{int} C_+$.

Similarly, working with φ_-, we obtain another constant sign smooth solution $v_0 \in -\operatorname{int} C_+$. □

Next, using the Morse theory (critical groups), we will produce a third nontrivial smooth solution. To this end, first we compute the critical groups of $\varphi_\pm$ at infinity (see also Wang [1] and Jiang [2]).

Proposition 3.6. *If hypotheses $H(a)$ and $H(f)$ hold, then*

$$C_k(\varphi_\pm, \infty) = 0 \quad \forall k \geq 0. \tag{3.55}$$

Proof. We do the proof for φ_+, the proof for φ_- being similar.

By virtue of hypotheses $H(f)$(i) and (ii), for a given $\xi > 0$, we can find $c_{16} = c_{16}(\xi) > 0$, such that

$$F_+(z,\zeta) \geq \xi(\zeta^+)^p - c_{16} \quad \text{for almost all } z \in \Omega, \text{ all } \zeta \in \mathbb{R}. \tag{3.56}$$

Let

$$E_+ = \{u \in \partial B_1 : \ u^+ \neq 0\}, \tag{3.57}$$

where

$$\partial B_1 = \left\{ u \in W_0^{1,p}(\Omega) : \ \|u\| = 1 \right\}. \tag{3.58}$$

For $u \in E_+$ and $t > 0$, we have

$$\begin{aligned} \varphi_+(tu) &= \int_\Omega G(t\nabla u)dz - \int_\Omega F_+(z,tu)dz \\ &\leq c_{17}(1+t^p) - \xi t^p \|u^+\|_p^p - c_{16}|\Omega|_N \\ &= t^p\left(c_{17} - \xi\|u^+\|_p^p\right) + c_{17} + c_{16}|\Omega|_N \end{aligned} \tag{3.59}$$

for some $c_{17} > 0$ (see Corollary 2.4, (3.56) and recall that $\|u\| = \|\nabla u\|_p = 1$).

Choosing $\xi > c_{17}/\|u^+\|_p^p$, we see that

$$\varphi_+(tu) \longrightarrow -\infty \quad \text{as } t \longrightarrow +\infty. \tag{3.60}$$

Hypothesis $H(f)$(iii) implies that we can find $\beta_1 \in (0, \beta_0)$ and $M_7 > 0$, such that

$$f_+(z,\zeta)\zeta - pF(z,\zeta) \geq \beta_1 \zeta^\tau \quad \text{for almost all } z \in \Omega, \ \text{all } \zeta \geq M_7. \tag{3.61}$$

Then for all $y \in W_0^{1,p}(\Omega)$, we have

$$\int_\Omega (pF_+(z,y) - f_+(z,y)y)dz \leq -\int_{\{y \geq M_7\}} \beta_1 y^\tau \, dz + c_{18}, \tag{3.62}$$

for some $c_{18} > 0$ (see (3.61)). Let $c_{19} = c_{18} + c_4|\Omega|_N > 0$ (see hypothesis $H(a)$(iv)) and choose $\gamma < -c_{19}$. Because of (3.60), for $u \in E_+$ and for large $t > 0$, we have

$$\varphi_+(tu) = \int_\Omega G(t\nabla u)dz - \int_\Omega F_+(z,tu)dz \leq \gamma. \tag{3.63}$$

Since $\varphi_+(0) = 0$, we can find $t^* > 0$, such that

$$\varphi_+(t^* u) = \gamma. \tag{3.64}$$

Also, for large $t > 0$, we have

$$\begin{aligned}
\frac{d}{dt}\varphi_+(tu) &= \langle \varphi'_+(tu),\ u\rangle \\
&= \frac{1}{t}\langle \varphi'_+(tu),\ tu\rangle \\
&= \frac{1}{t}\left(\int_\Omega (a(t\nabla u),\ t\nabla u)_{\mathbb{R}^N}\, dz - \int_\Omega f_+(z,tu)tu\, dz\right) \\
&\leq \frac{1}{t}\left(\int_\Omega pG(t\nabla u)dz + c_4|\Omega|_N - \int_\Omega pF_+(z,tu)dz + c_{18}\right) \\
&\leq \frac{1}{t}\left(\gamma + c_{18} + c_4|\Omega|_n\right) \\
&< 0
\end{aligned} \tag{3.65}$$

(see hypothesis $H(a)(\mathrm{iv})$, (3.62), (3.63), and recall that $p\varphi_+(tu) \leq \varphi_+(tu) \leq \gamma < 0$). Hence, by the implicit function theorem, t^* is unique and in fact there is a unique function $\mu_+ \in C(E_+)$, such that

$$\varphi_+\left(\mu_+(u)u\right) = \gamma \quad \forall u \in E_+. \tag{3.66}$$

Let

$$D_+ = \left\{u \in W_0^{1,p}(\Omega) : u^+ \neq 0\right\}. \tag{3.67}$$

We set

$$\widehat{\mu}_+(u) = \frac{1}{\|u\|}\mu_+\left(\frac{u}{\|u\|}\right) \quad \forall u \in D_+. \tag{3.68}$$

Then $\widehat{\mu}_+ \in C(D_+)$ and

$$\varphi_+\left(\widehat{\mu}_+(u)u\right) = \gamma \quad \forall u \in D_+. \tag{3.69}$$

Moreover, if $\varphi_+(u) = \gamma$, then $\widehat{\mu}_+(u) = 1$. We set

$$\widetilde{\mu}_+(u) = \begin{cases} 1 & \text{if } \varphi_+(u) \leq \gamma, \\ \widehat{\mu}_+(u) & \text{if } \varphi_+(u) > \gamma. \end{cases} \tag{3.70}$$

Evidently $\widehat{\mu}_+ \in C(E_+)$. Let $h_+ : [0,1] \times D_+ \to D_+$ be defined by

$$h_+(t,u) = (1-t)u + t\widetilde{\mu}_+(u)u. \tag{3.71}$$

Clearly h_+ is continuous and

$$h_+(0,u) = u, \qquad h_+(1,u) = \tilde{\mu}_+(u)u \in \varphi_+^{\gamma}, \\ h_+(t,u) = u \quad \forall t \in [0,1], \ u \in \varphi_+^{\gamma} \tag{3.72}$$

(see (3.70)) and so

$$\varphi_+^{\gamma} \text{ is a strict deformation retract of } D_+. \tag{3.73}$$

It is easy to see that D_+ is contractible in itself. Hence

$$H_k\left(W_0^{1,p}(\Omega),\ D_+\right) = 0 \quad \forall k \geq 0 \tag{3.74}$$

(see Granas and Dugundji [16, p. 389]), so

$$H_k\left(W_0^{1,p}(\Omega),\ \varphi_+^{\gamma}\right) = 0 \quad \forall k \geq 0 \tag{3.75}$$

(see (3.73)) and thus

$$C_k(\varphi_+, \infty) = 0 \quad \forall k \geq 0 \tag{3.76}$$

(choosing $\gamma < 0$ negative enough).

The same applies for φ_-, using this time the sets

$$E_- = \{u \in \partial B_1 : u^- \neq 0\}, \\ D_- = \left\{u \in W_0^{1,p}(\Omega) : u^- \neq 0\right\}. \tag{3.77}$$

□

With suitable changes in the above proof, we can have the following result.

Proposition 3.7. *If hypotheses $H(a)$ and $H(f)$ hold, then*

$$C_k(\varphi, \infty) = 0 \quad \forall k \geq 0. \tag{3.78}$$

Proof. As before, hypotheses $H(f)$(i) and (ii) imply that for a given $\xi > 0$, we can find $c_{20} = c_{20}(\xi) > 0$, such that

$$F(z,\zeta) \ \geq \ \xi|\zeta|^p - c_{20} \quad \text{for almost all } z \in \Omega, \ \text{all } \zeta \in \mathbb{R}. \tag{3.79}$$

Let $u \in \partial B_1 = \{u \in W_0^{1,p}(\Omega) : \|u\| = 1\}$ and $t > 0$. Then

$$\begin{aligned}
\varphi(tu) &= \int_\Omega G(t\nabla u)dz - \int_\Omega F(z,tu)dz \\
&\le c_{21}(1+t^p) - \xi t^p \|u\|_p^p + c_{20}|\Omega|_N \\
&\le t^p \left(c_{21} - \xi\|u\|_p^p\right) + c_{21} + c_{20}|\Omega|_N
\end{aligned} \tag{3.80}$$

for some $c_{21} > 0$ (see Corollary 2.4, (3.79) and recall that $\|u\| = 1$). Choosing $\xi > c_{21}/\|u\|_p^p$, we see that

$$\varphi(tu) \longrightarrow -\infty \quad \text{as } t \longrightarrow +\infty. \tag{3.81}$$

Hypothesis $H(f)$(iii) implies that we can find $\beta_1 \in (0, \beta_0)$ and $M_8 > 0$, such that

$$f(z,\zeta)\zeta - pF(z,\zeta) \ \ge \beta_1|\zeta|^\tau \quad \text{for almost all } z \in \Omega, \ \text{all } |\zeta| \ge M_8. \tag{3.82}$$

Then, for any $y \in W_0^{1,p}(\Omega)$, we have

$$\begin{aligned}
\int_\Omega (pF(z,y) - f(z,y)y)d &= \int_{\{|y|<M_8\}} (pF(z,y) - f(z,y)y)dz + \int_{\{|y|\ge M_8\}} (pF(z,y) - f(z,y)y)dz \\
&\le -\int_{\{|y|\ge M_8\}} \beta|y|^\tau\, dz + c_{22},
\end{aligned} \tag{3.83}$$

for some $c_{22} > 0$ (see hypothesis $H(f)$(i)).

Let $c_{23} = c_{22} + c_4|\Omega|_N$ (see hypothesis $H(a)$(iv)) and choose $\gamma < -c_{23}$. Because of (3.81), for a given $u \in \partial B_1$ and for large $t > 0$, we have

$$\varphi(tu) \ = \int_\Omega G(t\nabla u)dz - \int_\Omega F(z,tu)dz \ \le \gamma. \tag{3.84}$$

We also have

$$\begin{aligned}
\frac{d}{dt}\varphi(tu) &= \langle \varphi'(tu),\ u\rangle \\
&= \frac{1}{t}\langle \varphi'(tu),\ tu\rangle \\
&= \frac{1}{t}\left(\int_\Omega (a(t\nabla u),\ t\nabla u)_{\mathbb{R}^N} - \int_\Omega f(z,tu)tu\,dz\right) \\
&\leq \frac{1}{t}\left(\int_\Omega pG(t\nabla u)\ dz + c_4|\Omega|_N - \int_\Omega pF(z,tu)dz + c_{22}\right) \\
&\leq \frac{1}{t}\left(\gamma + c_{22} + c_4|\Omega|_N\right) \\
&< 0
\end{aligned} \tag{3.85}$$

(see hypothesis $H(f)$(iv), (3.83), (3.84) and note that $p\varphi(tu) \leq \varphi(tu) \leq \gamma < 0$).

The implicit function theorem implies that there exists unique $\mu \in C(\partial B_1)$, such that

$$\varphi(\mu(u)u) = \gamma \quad \forall u \in \partial B_1. \tag{3.86}$$

We define

$$\widehat{\mu}(u) = \frac{1}{\|u\|}\mu\left(\frac{u}{\|u\|}\right) \quad \forall u \neq 0. \tag{3.87}$$

Then $\widehat{\mu} \in C(W_0^{1,p}(\Omega) \setminus \{0\})$ and

$$\varphi(\widehat{\mu}(u)u) = \gamma \quad \forall u \in W_0^{1,p}(\Omega) \setminus \{0\}. \tag{3.88}$$

Moreover, if $\varphi(u) = \gamma$, then $\widehat{\mu}(u) = 1$. We introduce

$$\widetilde{\mu}(u) = \begin{cases} 1 & \text{if } \varphi(u) \leq \gamma, \\ \widehat{\mu}(u) & \text{if } \varphi(u) > \gamma. \end{cases} \tag{3.89}$$

Evidently $\widetilde{u} \in C(W_0^{1,p}(\Omega) \setminus \{0\})$. Let $h : [0,1] \times (W_0^{1,p}(\Omega) \setminus \{0\}) \to W_0^{1,p}(\Omega) \setminus \{0\}$ be defined by

$$h(t,u) = (1-t)u + t\widetilde{\mu}(u)u. \tag{3.90}$$

Then

$$\begin{gathered}
h(0,u) = u, \qquad h(1,u) = \widetilde{\mu}(u)u \in \varphi^\gamma, \\
h(t,u) = u \quad \forall t \in [0,1],\ u \in \varphi^\gamma,
\end{gathered} \tag{3.91}$$

so

$$\varphi^{\gamma} \text{ is a strong deformation retract of } W_0^{1,p}(\Omega) \setminus \{0\}. \tag{3.92}$$

Via the radial retraction, we see that ∂B_1 is a retract of $W_0^{1,p}(\Omega) \setminus \{0\}$ and $W_0^{1,p}(\Omega) \setminus \{0\}$ is deformable into ∂B_1. Invoking Theorem 6.5 of Dugundji [17, p. 325], we have that

$$\partial B_1 \text{ is a deformation retract of } W_0^{1,p}(\Omega) \setminus \{0\}, \tag{3.93}$$

so

$$\partial B_1 \text{ and } \varphi^{\gamma} \text{ are homotopy equivalent.} \tag{3.94}$$

Thus

$$H_k\left(W_0^{1,p}(\Omega),\ \partial B_1\right) = H_k\left(W_0^{1,p}(\Omega),\ \varphi^{\gamma}\right) \quad \forall k \geq 0, \tag{3.95}$$

and finally

$$C_k(\varphi, \infty) = H_k\left(W_0^{1,p}(\Omega),\ \partial B_1\right) \quad \forall k \geq 0. \tag{3.96}$$

But ∂B_1 is an absolute retract of $W_0^{1,p}(\Omega)$ (see, e.g., Gasiński and Papageorgiou [3, p. 691]), hence contractible in itself. Therefore

$$H_k\left(W_0^{1,p}(\Omega),\ \partial B_1\right) = 0 \quad \forall k \geq 0, \tag{3.97}$$

so

$$C_k(\varphi, \infty) = 0 \quad \forall k \geq 0. \tag{3.98}$$

□

Now we are ready to produce the third nontrivial smooth solution for problem (1.5).

Theorem 3.8. *If hypotheses $H(a)$ and $H(f)$ hold, then problem* (1.5) *has at least three nontrivial smooth solutions*

$$u_0 \in \operatorname{int} C_+, \qquad v_0 \in -\operatorname{int} C_+, \qquad y_0 \in C_0^1\left(\overline{\Omega}\right) \setminus \{0\}. \tag{3.99}$$

Proof. From Proposition 3.5, we already have two nontrivial constant sign and smooth solutions

$$u_0 \in \operatorname{int} C_+, \qquad v_0 \in -\operatorname{int} C_+. \tag{3.100}$$

We assume that

$$K_\varphi = \{0, u_0, v_0\}. \tag{3.101}$$

Otherwise we already have a third nontrivial solution y_0, and by the nonlinear regularity theory, $y_0 \in C_0^1(\overline{\Omega})$, so we completed the proof.

Claim 1. $C_k(\varphi_+, u_0) = C_k(\varphi_-, v_0) = \delta_{k,1}\mathbb{Z}$ for all $k \geq 0$.

We do the proof for the pair (φ_+, u_0), the proof for the pair (φ_-, v_0) being similar.

We start by noting that $K_{\varphi_+} = \{0, u_0\}$. Indeed, suppose that $u \in K_{\varphi_+} \setminus \{0\}$. Then

$$A(u) = N_{f_+}(u). \tag{3.102}$$

Acting with $-u^- \in W_0^{1,p}(\Omega)$, we obtain

$$\frac{c_1}{p-1}\|\nabla u^-\|_p^p \leq 0 \tag{3.103}$$

(see Lemma 2.3(c)) and so $u_0 \geq 0$, $u_0 \neq 0$. Moreover, by nonlinear regularity (see Ladyzhenskaya and Uraltseva [13] and Lieberman [14]), we have that $u \in C_+ \setminus \{0\}$. Since $\varphi'|_{C_+} = \varphi'_+|_{C_+}$, we infer that $u \in K_\varphi = \{0, u_0, v_0\}$, and hence $u = u_0$.

Choose $\gamma, \vartheta \in \mathbb{R}$, such that

$$\gamma < 0 = \varphi_+(0) < \vartheta < \eta_+ \leq \varphi_+(u_0) \tag{3.104}$$

(see (3.47)), and consider the following set:

$$\varphi_+^\gamma \subseteq \varphi_+^\vartheta \subseteq W_0^{1,p}(\Omega) = W. \tag{3.105}$$

We consider the long exact sequence of singular homology groups corresponding to the above triple. We have

$$\cdots \longrightarrow H_k\left(W, \varphi_+^\gamma\right) \xrightarrow{i_*} H_k\left(W, \varphi_+^\vartheta\right) \xrightarrow{\partial_*} H_{k-1}\left(\varphi_+^\vartheta, \varphi_+^\gamma\right) \longrightarrow \cdots \quad \forall k \geq 0, \tag{3.106}$$

where i_* is the group homomorphism induced by the embedding $i : \varphi_+^\gamma \to \varphi_+^\vartheta$ and ∂_* is the boundary homomorphism. Note that

$$H_k\left(W, \varphi_+^\gamma\right) = C_k(\varphi_+, \infty) \quad \forall k \geq 0 \tag{3.107}$$

(since $\gamma < 0$, $K_{\varphi_+} = \{0, u_0\}$ and $0 = \varphi_+(0) < \eta_+ \leq \varphi_+(u_0)$; see (3.47)), so

$$H_k\left(W, \varphi_+^\gamma\right) = 0 \quad \forall k \geq 0 \tag{3.108}$$

(see Proposition 3.6).

From the choice of $\vartheta > 0$, the only critical value of φ_+ in the interval (γ, ϑ) is 0. Hence

$$H_{k-1}\left(\varphi_+^{\vartheta}, \varphi_+^{\gamma}\right) = C_{k-1}(\varphi_+, 0) = \delta_{k-1,0}\mathbb{Z} = \delta_{k,1}\mathbb{Z} \quad \forall k \geq 0 \tag{3.109}$$

(see Proposition 3.3).

Finally, for the same reason, we have

$$H_k\left(W, \varphi_+^{\vartheta}\right) = C_k(\varphi_+, u_0) \quad \forall k \geq 0. \tag{3.110}$$

From (3.108), (3.109), and (3.110), it follows that in (3.106) only the tail (i.e., $k = 1$) is nontrivial. The rank theorem implies that

$$\begin{aligned} \text{rank } H_1\left(W, \varphi_+^{\vartheta}\right) &= \text{rank im } \partial_* + \text{rank ker } \partial_* \\ &= \text{rank im } \partial_* + \text{rank im } i_* \\ &= 1 + 0 = 1 \end{aligned} \tag{3.111}$$

(by virtue of the exactness of (3.106)), so

$$\text{rank } C_1(\varphi_+, u_0) \leq 1 \tag{3.112}$$

(see (3.110)).

But u_0 is a critical point of mountain pass type for φ_+. Hence

$$\text{rank } C_1(\varphi_+, u_0) \geq 1. \tag{3.113}$$

From (3.112) and (3.113) and since

$$C_k(\varphi_+, u_0) = H_k\left(W, \varphi_+^{\vartheta}\right) = 0 \quad \forall k \neq 1, \tag{3.114}$$

we infer that

$$C_k(\varphi_+, u_0) = \delta_{k,1}\mathbb{Z} \quad \forall k \geq 0. \tag{3.115}$$

Similarly, we show that

$$C_k(\varphi_-, v_0) = \delta_{k,1}\mathbb{Z} \quad \forall k \geq 0. \tag{3.116}$$

This proves Claim 1.

Claim 2. $C_k(\varphi, u_0) = C_k(\varphi_+, u_0)$, $C_k(\varphi, v_0) = C_k(\varphi_-, v_0)$ for all $k \geq 0$.

We consider the homotopy

$$h_1(t,u) = (1-t)\varphi_+(u) + t\varphi(u) \quad \forall (t,u) \in [0,1] \times W_0^{1,p}(\Omega). \tag{3.117}$$

Clearly $u_0 \in K_{h(t,\cdot)}$ for all $t \in [0,1]$. We will show that there exists $\varrho > 0$, such that

$$B_\varrho(u_0) \cap K_{h_1(t,\cdot)} = \{u_0\} \quad \forall t \in [0,1], \tag{3.118}$$

where

$$B_\varrho(u_0) = \left\{u \in W_0^{1,p}(\Omega) : \|u-u_0\| < \varrho\right\}. \tag{3.119}$$

Arguing by contradiction, suppose that (3.118) is not true for any $\varrho > 0$. Then we can find two sequences $\{t_n\}_{n\ge1} \subseteq [0,1]$ and $\{u_n\}_{n\ge1} \subseteq W_0^{1,p}(\Omega) \setminus \{u_0\}$, such that

$$\begin{aligned} &t_n \longrightarrow t \quad \text{in } [0,1], \\ &u_n \longrightarrow u_0 \quad \text{in } W_0^{1,p}(\Omega), \\ &(h_{t_n})'(u_n) = 0 \quad \forall n \ge 1. \end{aligned} \tag{3.120}$$

For every $n \ge 1$, we have

$$A(u_n) = (1-t_n)N_{f_+}(u_n) + t_n N_f(u_n), \tag{3.121}$$

so

$$\begin{aligned} &-\operatorname{div} a(\nabla u_n(z)) = (1-t_n)f(z,u_n^+(z)) + t_n f(z,u_n(z)) \quad \text{a.e. in } \Omega, \\ &u_n|_{\partial\Omega} = 0. \end{aligned} \tag{3.122}$$

From Ladyzhenskaya and Uraltseva [13, p. 286], we know that we can find $M_9 > 0$, such that

$$\|u_n\|_\infty \le M_9 \quad \forall n \ge 1. \tag{3.123}$$

Then from Lieberman [14, p. 320], we infer that there exist $\alpha \in (0,1)$ and $M_{10} > 0$, such that

$$u_n \in C_0^{1,\alpha}\left(\overline{\Omega}\right), \quad \|u_n\|_{C_0^{1,\alpha}(\overline{\Omega})} \le M_{10} \quad \forall n \ge 1. \tag{3.124}$$

The compactness of the embedding $C_0^{1,\alpha}(\overline{\Omega})$ into $C_0^1(\overline{\Omega})$ and (3.120) imply that

$$u_n \longrightarrow u_0 \quad \text{in } C_0^1\left(\overline{\Omega}\right), \tag{3.125}$$

so, there exists $n_0 \geq 1$, such that

$$u_n \in \text{int } C_+ \quad \forall n \geq n_0 \tag{3.126}$$

(recall that $u_0 \in \text{int } C_+$; see Proposition 3.5) and thus $\{u_n\}_{n \geq n_0} \subseteq K_{\varphi_+}$ are distinct solutions of (1.5), a contradiction.

Therefore (3.118) holds for some $\varrho > 0$. Invoking the homotopy invariance property of critical groups, we have

$$C_k(\varphi, u_0) = C_k(\varphi_+, u_0) \quad \forall k \geq 0. \tag{3.127}$$

Similarly, we show that

$$C_k(\varphi, v_0) = C_k(\varphi_-, v_0) \quad \forall k \geq 0. \tag{3.128}$$

This proves Claim 2.

From Claims 1 and 2, we have that

$$C_k(\varphi, u_0) = C_k(\varphi, v_0) = \delta_{k,1}\mathbb{Z} \quad \forall k \geq 0. \tag{3.129}$$

Also, we have

$$C_k(\varphi, 0) = \delta_{k,0}\mathbb{Z} \quad \forall k \geq 0 \tag{3.130}$$

(see Proposition 3.3) and

$$C_k(\varphi, \infty) = 0 \quad \forall k \geq 0 \tag{3.131}$$

(see Proposition 3.7).

Since $K_\varphi = \{0, u_0, v_0\}$, from (3.129), (3.130), (3.131), and the Morse relation (2.13) with $t = -1$, we have

$$2(-1)^1 + (-1)^0 = 0, \tag{3.132}$$

a contradiction.

Therefore, there exists $y_0 \in K_\varphi$, $y_0 \notin \{0, u_0, v_0\}$. So, y_0 solves (1.5), and by the nonlinear regularity theory, $y_0 \in C_0^1(\overline{\Omega})$. □

Remark 3.9. Even in the Hilbert space case (i.e., $p = 2$), our result is more general than that of Wang [1], since we go beyond the Laplace differential operator and our hypotheses on the reaction f are considerably more general.

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